Cosmología / Cosmología Observacional, lecture 16

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Statistical measures (1)

- With a mean galaxy density $\bar{n}$, the probability to find a galaxy in a volume dV is given as

$$P_1 = \bar{n} dV. \tag{1}$$

- The probability of finding a galaxy in dV at location $\vec{x}$ and another one at $\vec{y}$ is given as

$$P_2 = (\bar{n} dV)^2 [1 + \xi_g(\vec{x}, \vec{y})]. \tag{2}$$

- If the galaxy distribution is uncorrelated, we would expect $\xi = 0$, i.e. $P_2 = P_1^2$.
- We can also define the **correlation function** for the total matter density as

$$\begin{aligned} \langle \rho(\vec{x}) \rho(\vec{y}) \rangle &= \bar{\rho}^2 \langle [1 + \delta(\vec{x})][1 + \delta(\vec{y})] \rangle & (3) \\ &= \bar{\rho}^2 (1 + \langle \delta(\vec{x}) \delta(\vec{y}) \rangle) & (4) \\ &= \bar{\rho}^2 [1 + \xi(\vec{x}, \vec{y})]. & (5) \end{aligned}$$

Statistical measures (2)

- In a statistically homogeneous Universe, the correlation function ξ can only depend on $|\vec{x} - \vec{y}|$, but not on the direction of $\vec{x} - \vec{y}$.
- For a homogeneous random field, the ensemble average $\langle\rangle$ can further be replaced by a spatial average.
- The latter is the only way to measure the correlation function, as there is only one realization of our Universe.
- From galaxy surveys, we have

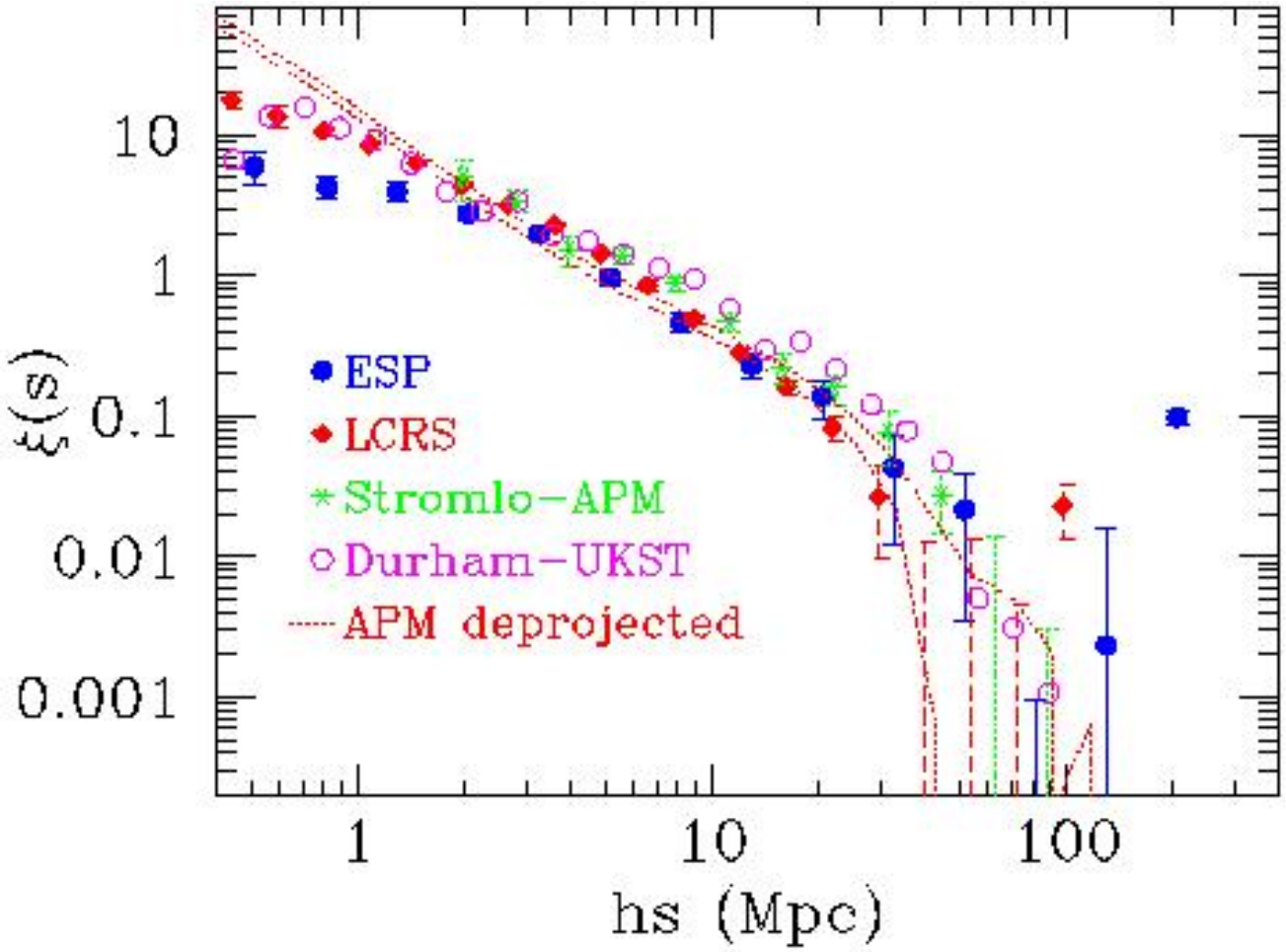
$$\xi_g(r) = \left(\frac{r}{r_0}\right)^{-\gamma} \tag{6}$$

with $r_0 \sim 5h^{-1}$ Mpc with $\gamma \sim 1.8$.

- The relation holds on scales of

$$2h^{-1}\ \mathrm{Mpc} \leq r \leq 30h^{-1}\ \mathrm{Mpc}. \tag{7}$$

Statistical measures (3)



Observed correlation function from galaxy surveys.

Statistical measures (4)

- An equivalent description can be obtained by considering the amount of structure on a given scale $L = \frac{2\pi}{k}$.
- The power spectrum $P(k)$ can be obtained from the correlation function $\xi(r)$ via a Fourier transform

$$P(k) = 2\pi \int_0^\infty dr\, r^2 \frac{\sin(kr)}{kr} \xi(r). \tag{8}$$

- The power-spectrum describes the second-order statistics of a random field, but in the general case, higher-order correlations may exist.
- Inflation is however expected to produce a **Gaussian random field**, which is fully described by second-order statistics.
- Indeed, the **Planck satellite** recently confirmed the absence of non-Gaussianities.

Statistical measures (5)

- As fluctuations grow, both the power spectrum $P(k)$ and the correlation function $\xi(r)$ are expected to evolve with redshift.
- We recall that

$$\delta(\vec{x}, t) = D_+(t)\delta_0(\vec{x}), \qquad D_+(t) \propto a. \tag{9}$$

- Using

$$\xi(\vec{x}, \vec{y}) = \langle \delta(\vec{x})\delta(\vec{y}) \rangle, \tag{10}$$

we have

$$\xi(\vec{x}, t) = D_+^2(t)\xi(\vec{x}, t_0) \tag{11}$$

and accordingly

$$P(k, t) = D_+^2(t)P(k, t_0) = D_+^2(t)P_0(k), \tag{12}$$

where k is a co-moving wave number and P_0 the initial power spectrum.

The initial power spectrum (1)

- In the radiation-dominated era, the expansion of the Universe followed a power-law as $a(t) \propto t^{1/2}$.
- In the absence of a characteristic length scale, the power spectrum should follow a power-law as

$$P(k) \propto k^{n_s}. \tag{13}$$

- Based on scaling relations, Harrison, Zeldovich & Peoples argued for $n_s = 1$.
- Indeed, inflationary models predict $n_s \lesssim 1$ ($n_s = 1$ if inflation continues forever).
- With such an initial spectrum, we would obtain a fluctuation spectrum

$$P(k, t_i) = D_+^2(t_i) A k^{n_s}, \tag{14}$$

where A is a normalization constant.

The initial power spectrum (2)

- For the evolution of density perturbations, we so far assumed Newtonian theory.
- Our calculation of the growth factor was further valid only in the matter-dominated era.
- In a radiation-dominated Universe, the evolution however is different, and depends on the relative size of perturbations compared to the horizon.
- As a result, a correction term of the form

$$P_0(k) = Ak^{n_s} T^2(k) \tag{15}$$

needs to be introduced. $T(k)$ is the **transfer function**.

The initial power spectrum (3)

- In the radiation-dominated era, the co-moving horizon is given as

$$r_{H,com}(z) \sim \frac{c}{H_0\sqrt{\Omega_r}}\frac{1}{1+z}. \tag{16}$$

- A fluctuation of co-moving size L enters the horizon when

$$r_{H,com}(z_{enter}) = L. \tag{17}$$

- Relativistic perturbation theory shows that fluctuations of size L grow as a^2 for $L > r_{H,com}$ in the radiation-dominated era.
- After entering the horizon, the growth of perturbations essentially stalls as a result of radiation pressure (until the matter-dominated era).

The initial power spectrum (4)

- If a perturbation of size L enters the horizon during the matter-dominated phase, it thus cannot grow for

$$z_{eq} \leq z \leq z_{enter}(L). \tag{18}$$

- The length scale L_0 with

$$z_{eq} = z_{enter}(L_0) \tag{19}$$

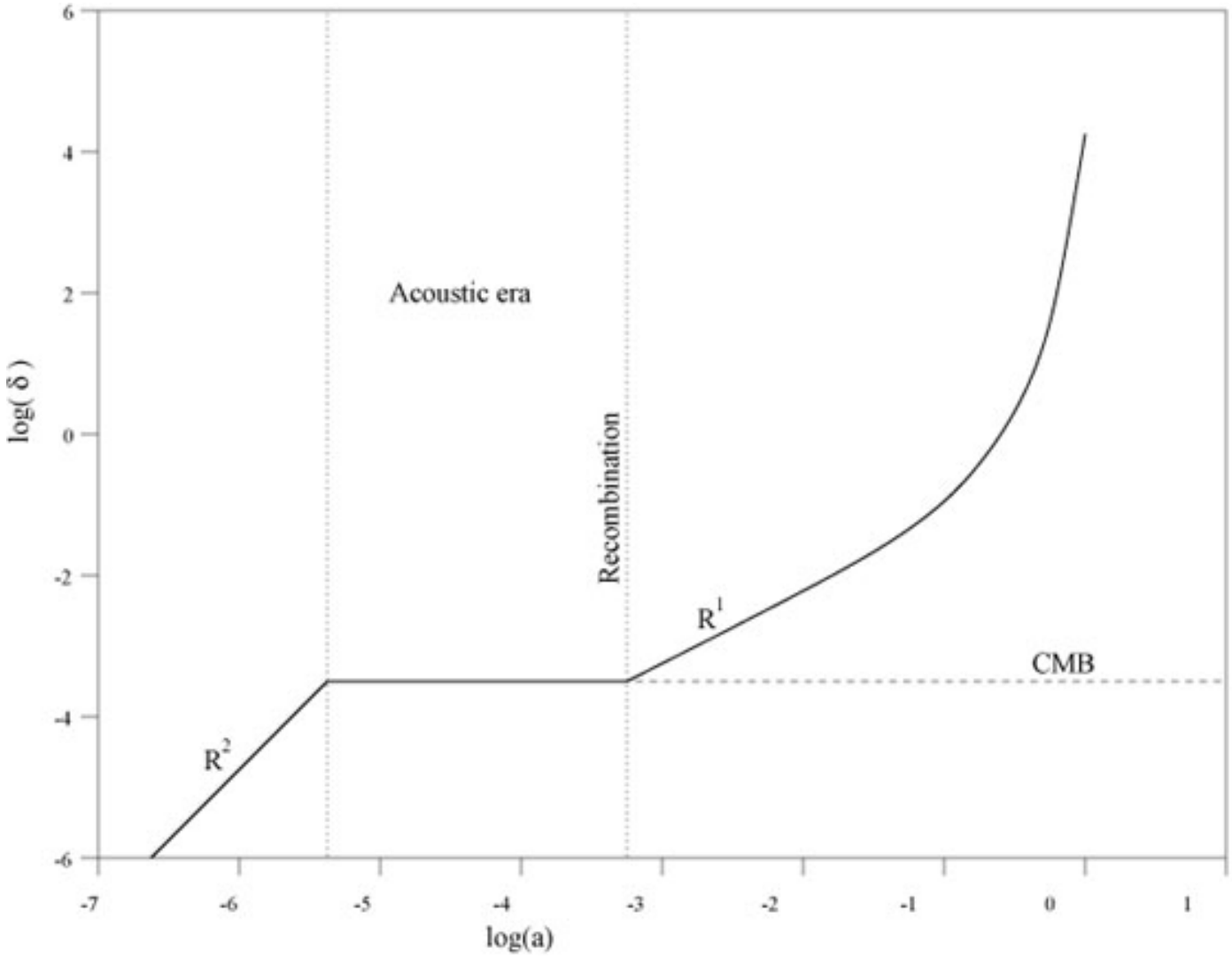
or

$$\begin{aligned} L_0 &= r_{H,com}(z_{eq}) = \frac{c}{\sqrt{2}H_0}\frac{1}{\sqrt{(1+z_{eq})\Omega_m}} \\ &= \frac{c}{\sqrt{2}H_0}\frac{1}{\sqrt{23900\Omega_m h}} \sim 12(\Omega_m h^2)^{-1}\ \mathrm{Mpc}. \end{aligned} \tag{20}$$

is the smallest length scale for which no suppression occurs.

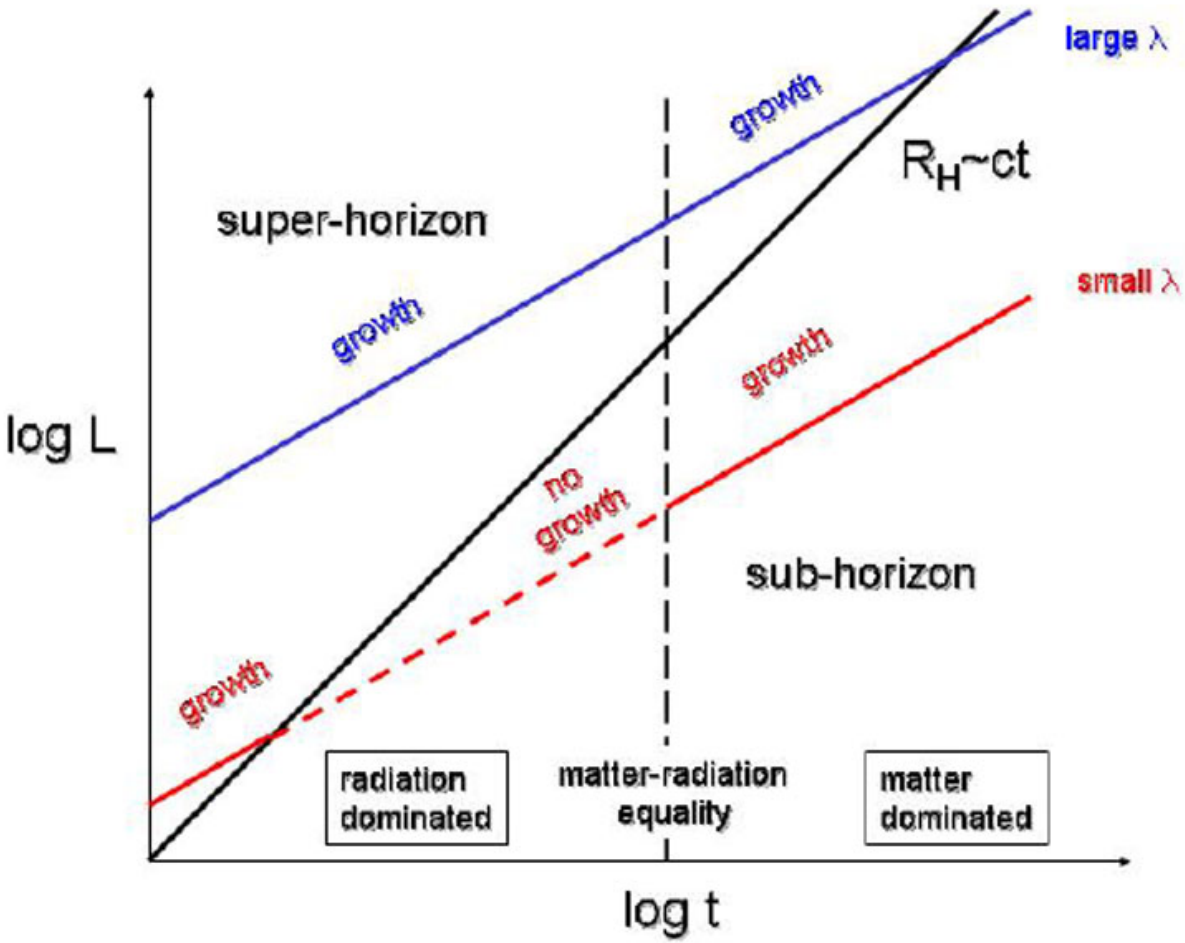
- Fluctuations with $L > L_0$ enter the horizon during matter-domination; their growth is never impeded by radiation pressure.

The initial power spectrum (5)



Growth of density fluctuations.

The initial power spectrum (6)



Impact of the horizon on the growth of fluctuations in the radiation-dominated era.

The initial power spectrum (7)

- If fluctuations enter the horizon at $z_{enter} < z_{eq}$, their growth is inhibited by a factor

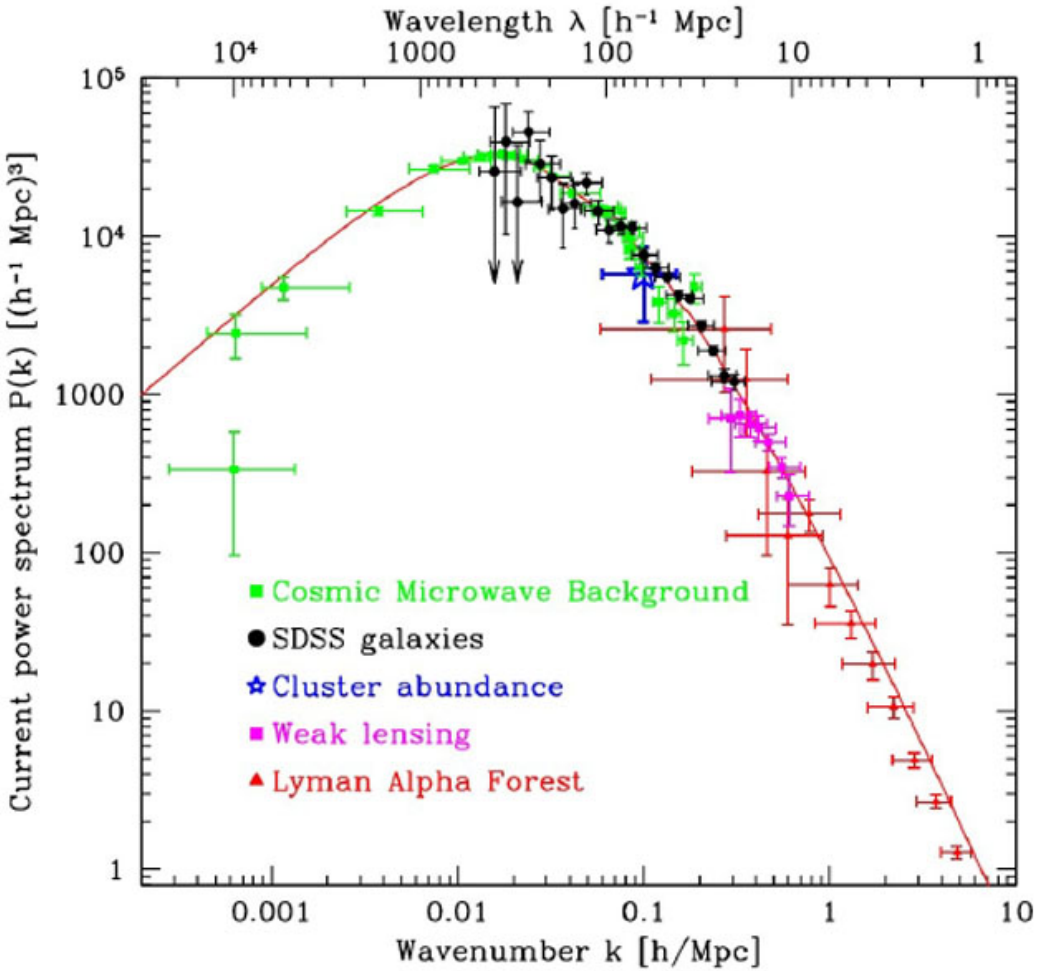
$$\left(\frac{a_{enter}}{a_{eq}}\right)^{-2} \tag{21}$$

  compared to fluctuations entering after z_{eq}.
- For $k \ll L_0^{-1}$, the transfer function is thus given as $T(k) \sim 1$.
- For $k \gg L_0^{-1}$,

$$T(k) \sim \left(\frac{a_{enter}}{a_{eq}}\right)^{-2} = (k\, r_{\mathrm{H,com}})^{-2}\,. \tag{22}$$

- Accurate solutions of the transfer function follow via numerical integration.

The initial power spectrum (8)



The resulting power spectrum.